Associative and commutative dg-algebras in positive characteristic

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Abstract

In this expository piece, we construct an example in characteristic p of two commutative dg-algebras which are quasi-isomorphic as associative but not commutative dg-algebras.[1]

1 Introduction

This goal of this article is to explore a recent problem in homotopy theory. To start with, we explain the background, this part is self-contained and should be accessible to any working mathematician. Then we shall provide a sketch as to how to solve it. The approach used is surprisingly not very technically sophisticated; instead, we make use of elementary combinatorics, linear algebra and number theory.

To commence, we introduce some theory that almost every mathematician will be somewhat familiar with and more details can be found in a good homological algebra textbook: for example [5].

Definition 1.1. A chain complex over a field is a collection of vector spaces $A^i : i \in \mathbb{Z}$ and maps $d^n : A^n \to A^{n+1}$ such that $d^{n+1} \circ d^n = 0$ for all n.

$$\cdots \longrightarrow A^i \xrightarrow{d} A^{i+1} \xrightarrow{d} A^{i+2} \xrightarrow{d} A^{i+3} \xrightarrow{d} A^{i+4} \xrightarrow{d} \cdots$$

For notational convenience, we generally suppress the superscript n when the target and source of the map are implicitly understood. A map of chain complexes $A \to B$ is a collection of maps f_n that commute with the differential. In other words, the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & A^i \arrow[r, "d"] \arrow[d, "f"] & A^{i+1} \arrow[r, "d"] \arrow[d, "f"] & A^{i+2} \arrow[r, "d"] \arrow[d, "f"] & A^{i+3} \arrow[r, "d"] \arrow[d, "f"] & A^{i+4} \arrow[r, "d"] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & B^i \arrow[r, "d"] & B^{i+1} \arrow[r, "d"] & B^{i+2} \arrow[r, "d"] & B^{i+3} \arrow[r, "d"] & B^{i+4} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

A great example of this, and in fact the primordial one, is that if you have a smooth manifold M, the collection of differential forms on M form a chain complex $\Omega^\bullet(M)$ called *the de Rham forms*, where $\Omega^n(M)$ is the vector space of differential n-forms and the differential is the exterior derivative. The de Rham forms can be used to compute some interesting global homotopy invariants of manifolds.

Definition 1.2. The cohomology of a chain complex A is the following set of groups

$$H^i(A) = \ker d^i / \operatorname{Im} d^{i-1}.$$

where $\ker d^i$ is the kernel of d^i and $\operatorname{Im} d^{i-1}$ is the image of d^{i+1}. We refer to elements in $\ker d^i$ as *cocycles* and elements in $\operatorname{Im} d^{i-1}$ as *coboundaries.* A map of chain complexes $f : A \to B$ induces group homomorphisms

$$H(f^i) : H^i(A) \to H^i(B).$$

[1] We thank Chase Ford for helpful feedback on the first version of this manuscript

The remarkable thing about the cohomology groups of $\Omega^\bullet(M)$ is that they are a *homotopy invariant* of the manifold M. In other words, they are preserved by squashing and squeezing (but not cutting) it. A very similar invariant - *singular cohomology* - of more general topological spaces can be produced by probing a manifold with simplices of varying dimensions to produce the *singular cochain complex*.

The attentive reader probably notices an issue here: not all manifolds or topological spaces with the same cohomology groups are homotopy equivalent. For example, $S^1 \times S^1$, a torus, has the same cohomology groups as $S^2 \vee S^1 \vee S^1$, a sphere and two circles glued together at a point. It would be desirable to have finer invariants, capable of distinguishing more spaces.

As it turns out, the de Rham forms actually have a little more structure, a commutative operation called *wedge product* that behaves well with respect to the differential. The singular cochains have a similar operation induced by the diagonal map

$$X \to X \times X$$

$$x \mapsto (x, x)$$

Definition 1.3. An *associative dg-algebra* is a chain complex A equipped with a binary associative multiplication $- \cup - : A^p \otimes A^q \to A^{p+q}$ and d satisfies the Leibniz rule from differentiation ie.

$$d(x \cup y) = d(x) \cup y + (-1)^{|x|} x \cup d(y).$$

where $|x|$ is the degree of x. If the product is also graded commutative[2], ie.

$$x \cup y = (-1)^{|x||y|} y \cup x$$

we call A a *commutative dg-algebra*.

Remark 1.4. An associative algebra structure on a chain complex A induces a graded multiplication on the cohomology of A,

$$H^i(A) \otimes H^j(A) \to H^{i+j}(A)$$

on the cohomology groups, turning the direct sum $\bigoplus H^i(A)$ into a ring. If the algebra structure is commutative, then the ring will be commutative too.

Again, a delicate question is how one extracts a homotopy invariant from this algebra structure. To do this, we shall need to introduce the notion of *quasi-isomorphism*.

A quasi-isomorphism between chain complexes A to B is a map that

a) induces an isomorphism on cohomology.

b) preserves any underlying algebraic structure.

A small problem is that quasi-isomorphisms do not necessarily admit inverses.

Example 1.5. For example, we can consider the following differential graded algebras[3]

$$A : \mathbb{R} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{R}x \xrightarrow{0} \mathbb{R}y \xrightarrow{[y \mapsto x^2]} \mathbb{R}x^2 \xrightarrow{0} \mathbb{R}xy \xrightarrow{xy \mapsto x^3} \mathbb{R}x^3 \longrightarrow \cdots$$

$$B : \mathbb{R}z \xrightarrow{0} 0 \xrightarrow{0} \mathbb{R}x \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \cdots$$

To recap, A is the polynomial algebra $\mathbb{R}[x, y]/(y^2)$, where the degree of $|x| = 2$ and $|y| = 1$. This chain complex has a differential generated by the rules

$$dx = 0 \qquad y = x^2$$

[2] Note that a commutative algebra is also required to be associative.

[3] The reader well versed in rational homotopy theory may recognize A as the minimal model for S^2 and B as its cohomology algebra.

and extended via the Leibniz rule

$$d(xy) = d(x)y + (-1)^2 x d(y) = x^3.$$

The dg-algebra B is $\mathbb{R}[z]/(z^2)$, which is two dimensional. The cohomology of both of these complexes is

$$H^i(A) = H^i(B) = \begin{cases} \mathbb{R} & \text{if } i = 0, 2. \\ 0 & \text{otherwise.} \end{cases}$$

Elements of the cohomology group $H^2(A)$ are, by definition, equivalences classes of elements in $\ker d^2 \subseteq Rx$. In particular, a *cochain representative* for the unit in $H^2(A)$ is x, and a *cochain representative* for the unit in $H^2(B)$ is z. In general, there may be many such choices, but in this case, it is unique.

There is a quasi-isomorphism of commutative algebras from A to B given as follows.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "0"] \arrow[d, "\mathrm{id}"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{R}x \arrow[r, "0"] \arrow[d, "{[x \mapsto z]}"] & \mathbb{R}y \arrow[r, "{[y \mapsto x^2]}"] \arrow[d, "0"] & \mathbb{R}x^2 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{R}xy \arrow[r, "xy \mapsto x^3"] \arrow[d, "0"] & \mathbb{R}x^3 \arrow[r] \arrow[d, "0"] & \cdots \\
\mathbb{R} \arrow[r, "0"] & 0 \arrow[r, "0"] & \mathbb{R}x \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & \cdots
\end{tikzcd}$$

However, there is no quasi-isomorphism from $g : B \to A$. This is because z has to be sent to x, as a quasi-isomorphism is an isomorphism on cohomology groups. But then $g(0) = g(z^2) = g(z)^2 = x^2$ as g must preserve the algebraic structure. But g fails to be a linear map!

Quasi-isomorphism is transitive and reflexive, but as the previous example shows, it is unfortunately not symmetric. Therefore it is not an equivalence relation. But we can "pretend it is" by saying that chain complexes A and B are quasi-isomorphic as associative dg-algebras if there is a *zig-zag*

$$A = C_1 \xleftarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xleftarrow{f_3} \cdots \xleftarrow{f_{n-2}} C_{n-1} \xrightarrow{f_{n-1}} C_n = B.$$

The key point here is that each of the C_i is an associative dg-algebra and each quasi-isomorphism of associative algebras f_i, by definition, preserves the algebra structure. One way of thinking about what is happening here is that you are forcing the relation of "quasi-isomorphism" to be symmetric by adding formal inverses to each quasi-isomorphism.

So now, a natural question is to try to classify when are two associative dg-algebras A and B are quasi-isomorphic **as associative algebras**?

Example 1.6. A brief aside, if we work purely with chain complexes over a field, and assume no additional algebraic structure, there is always a quasi-isomorphism of chain complexes

$$A \xrightarrow{\sim} H(A).$$

So, if A has the same cohomology groups as B, we can always find a quasi-isomorphism

$$B \xrightarrow{\sim} H(B) = H(A) \xrightarrow{\sim} A.$$

Thus, two chain complexes have the same quasi-isomorphism type as chain complexes, if and only if, they have the same cohomology groups.

In light of Example 1.6, one might hope two associative dg-algebras A and B are quasi-isomorphic as associative (or commutative) algebras if and only they have the same cohomology ring. This turns out not to be the case, one can construct some concrete counterexamples.

When A and B are **commutative** and the zig-zag consists of **commutative** algebras, this turns out to be a very subtle question, which was elegantly answered by Sullivan [4] with his theory of minimal models, which completely answers the classification problem *rationally*. This is an amazing result, because it shows that the rational homotopy type of a topological space is completely determined by a (relatively!) small commutative dg-algebra. It formed part of the work that Sullivan was awarded an Abel prize for in 2022.

This all led Ricardo Campos, Dan Petersen, Daniel Robert-Nicoud, and Felix Wierstra to ask the following question [1]:

Question 1.7. If two commutative algebras A and B are quasi-isomorphic as **associative** algebras are they quasi-isomorphic as **commutative** algebras?

At first glance, the answer appears to be obviously no. Every zig-zag of commutative algebras

$$A = C_1 \xleftarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xleftarrow{f_3} \cdots \xleftarrow{f_{n-2}} C_{n-1} \xrightarrow{f_{n-1}} C_n = B.$$

is also a zig-zag of associative algebras, as commutative algebras are a special kind of associative algebra.

However, it is easy to find zig-zags

$$A = C_1 \xleftarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xleftarrow{f_3} \cdots \xleftarrow{f_{n-2}} C_{n-1} \xrightarrow{f_{n-1}} C_n = B.$$

where A and B are commutative, but the algebras C_i are associative but not commutative.

Example 1.8. There are some very simple examples of non-commutative, associative algebras that are weakly equivalent **as associative algebras** to commutative algebras. For a concrete example, consider the following algebra A, with a linear basis which has just five elements $\{x, y, xy, yx, z\}$, with the degrees given by $|x| = |y| = 2$ and $|z| = 3$ (in particular, there are some obvious relations here like $x^2 = y^2 = 0$). We set $dz = xy - yx$.

Now, consider B to be the commutative dg-algebra with a linear basis which has just three elements $\{a, b, ab\}$, with the degrees given by $|a| = |b| = 2$. There is a quasi-isomorphism from A to B as follows.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "\mathrm{id}"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{R}x \oplus \mathbb{R}y \arrow[r, "0"] \arrow[d, "{[x\mapsto a] \atop [y\mapsto b]}"] & \mathbb{R}z \arrow[r, "{[z\mapsto xy-yx]}"] \arrow[d, "0"] & \mathbb{R}xy \oplus \mathbb{R}yx \arrow[r, "0"] \arrow[d, "{[xy\mapsto ab] \atop [yx\mapsto -ab]}"] & 0 \arrow[r, "0"] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[r, "0"] \arrow[d, "0"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \cdots \\
0 \arrow[r, "0"] & 0 \arrow[r, "0"] & \mathbb{R}a \oplus \mathbb{R}b \arrow[r, "0"] & 0 \arrow[r, "0"] & \mathbb{R}ab \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & \cdots
\end{tikzcd}$$

There are thus many many more potential zig-zags in the larger category of associative dg algebras. However, if you work with commutative and associative algebras over a field of characteristic 0, they proved that the answer is actually yes [1]. In other words, if there is a zig-zag between two commutative algebras A and B where some of the C_i are associative but not necessarily commutative, there is an alternative zig-zag between A and B consisting entirely of commutative algebras.

The proof, as one might expect, is very technical and makes a lot of use of a collection of methods informally referred to as the *operadic calculus*.

However, this still leaves open the case of what happens when one works over a field of characteristic p.

Question 1.9. If two commutative algebras A and B over $\mathbb{F}_p$ are quasi-isomorphic as associative algebras are they quasi-isomorphic as commutative algebras?

Sadly and not completely unexpectedly, I should warn you that the answer here is no, but how to construct a counterexample? The construction actually ends up being mainly number theoretic and combinatorial in nature rather than homotopic.

The strategy here would be familiar to anyone who has ever sat a maths contest - one wants to look for homotopy invariants of commutative algebras that are not homotopy invariants of associative algebras. There is something very weird going on with the Frobenius map in characteristic p. Observe that, the Leibniz rule for differentiating tells us that in a commutative algebra A, we have:

$$d(x^p) = px^{p-1}dx = 0.$$

where the second equality holds because we are working over a field of characteristic p. In other words, x^p is always a cocycle. But this relation does not hold in a general associative algebra A. Instead, one has

$$d(x^p) = (dx)x^{p-1} + x(dx)x^{p-2} + \cdots + x^{p-1}dx.$$

One cannot rearrange the order of the multiplication in the terms, since A is not assumed to be associative. So this has no reason to a cocycle and generally will not be.

This suggests the following general strategy. You know that, if two commutative algebras A and B are associative quasi-isomorphic, they have the same cohomology ring and that this ring is even commutative.

Like any commutative ring, we have relations: $ab = 0$ for $a, b \in H^\bullet(A)$. Recall the definition of the cohomology groups $H^i(A) = \ker d^i / \operatorname{Im} d^{i-1}$.

This means that, on the level of A, a represents an equivalence class, and one may choose

$$\bar{a} \in \ker d^i \subset A^i.$$

representing it. Such choice may not unique, but given any two choices of representative $\bar{a}$ and $\bar{a}'$, they must be related by the rule

$$\bar{a} - \bar{a}' \in \operatorname{Im} d^{i-1}.$$

So, we may take representatives $\bar{a}, \bar{b} \in A$ ie. for $a, b \in H(A)$ in cohomology. The product $\bar{a}\bar{b}$ in A does not need to be 0 on the nose, but it does need to represent that equivalence class. So $\bar{a}\bar{b} \in \operatorname{Im} d$. It follows that one can find $\bar{c} \in A$ such that

$$\bar{a}\bar{b} = d\bar{c}.$$

Again, c is not necessarily unique, but the choice is sufficiently constrained as to still be useful. For example, you can add any cocycle σ to c and one still has

$$d(\bar{c} + \sigma) = \bar{a}\bar{b}$$

but things break if you add an element such that $d\sigma \neq 0$. Cocycles determine elements of the cohomology, so everything is well-defined in a quotient ring of the cohomology.

Now, you can just take the p^{th} power $\bar{c}^p$ and you get a cocycle and therefore an element in the cohomology. This is is not perfectly well defined, but, by keeping track of all our previous choices, you can show, without much difficulty, that it is well defined as an element in the quotient group

$$\frac{H(A)}{H(A)^p + a^p H(A) + b^p H(A)}.$$

Here, the $H(A)^p$ accounts for the ambiguity in the choice of $\bar{c}$, and $a^p H(A) + b^p H(A)$ accounts for the ambiguity in the choice of $\bar{a}$ and $\bar{b}$.

So now our strategy becomes clear. We just need to find a pair of commutative algebras A and B with the same cohomology, but where this invariant differs. Then we need to find an associative algebra C such that there is a zig-zag of associative weak equivalences

$$A \xleftarrow{\sim} C \xrightarrow{\sim} B$$

Building C, and ensuring it has the same cohomology ring as A and B, involves a little combinatorial trickery.But it's fundamentally just linear algebra - a concrete example can be found in the paper [2, Section 4.2.4]. There, as vector spaces, the commutative algebras A and B are 7 and 14 dimensional respectively, and the associative algebra C ends up having 32 basis elements.

This method of finding *higher invariants* that live just above the cohomology to solve very concrete problems has a rich history, going back to Massey [3], who used his eponymous products, constructed using similar vanishing arguments, to show that the Borromean rings were pairwise unlinked but cannot be separated.

References

[1] R. Campos, D. Petersen, D. Robert-Nicoud, and F. Wierstra. Lie, associative and commutative quasi-isomorphism. *Acta Math.*, 233(2):195–238, 2024.

[2] O. Flynn-Connolly. An obstruction theory for strictly commutative algebras in positive characteristic. *Algebraic & Geometric Topology*, To appear.

[3] W. S. Massey. Higher order linking numbers. In *Conf. on Algebraic Topology (Univ. of Illinois at Chicago Circle, Chicago, Ill., 1968)*, pages 174–205. Univ. of Illinois at Chicago Circle, Chicago, Ill., 1969.

[4] D. Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (47):269–331 (1978), 1977.

[5] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

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